

Extremes of functions with multiple variables (part II)

conditional extremes function

Here we have set apart the functions and conditions. Most often it is **one condition** , but in more serious instances could be two or more.

If we have function $f(x)$ and condition $\varphi(x)$, we first set up a function:

$$F(x) = f(x) + \lambda \cdot \varphi(x)$$

λ is an unknown coefficient that we ask:

- Find the first partial derivatives and equate them to zero
- Express x and y over λ
- Replace x and y in condition $\varphi(x)$ and find the value for λ (There may be several)
- Return that value in x and y , so we have stationary points
- Look for second total differential d^2F to examine whether it comes to maximum or minimum (If $d^2F < 0$ then it is max , and if $d^2F > 0$, then it is min.)

Example 1.

Find conditional extremes for function $z = ax + by$ if the condition is $x^2 + y^2 = 1$

Solution:

So, first set up a function:

$$F = ax + by + \lambda(x^2 + y^2 - 1) \quad \text{watch out, placed a condition equal to zero! } x^2 + y^2 = 1 \rightarrow \boxed{x^2 + y^2 - 1 = 0}$$

Still looking for first partial derivatives and equate them with 0.

$$F = ax + by + \lambda(x^2 + y^2 - 1)$$

$$\frac{\partial F}{\partial x} = a + 2\lambda x$$

$$\frac{\partial F}{\partial y} = a + 2\lambda y$$

$$a + 2\lambda x = 0 \rightarrow \boxed{x = -\frac{a}{2\lambda}}$$

$$b + 2\lambda y = 0 \rightarrow \boxed{y = -\frac{b}{2\lambda}}$$

These replace in condition to find a value for λ :

$$x^2 + y^2 = 1$$

$$\left(-\frac{a}{2\lambda}\right)^2 + \left(-\frac{b}{2\lambda}\right)^2 = 1$$

$$\frac{a^2}{4\lambda^2} + \frac{b^2}{4\lambda^2} = 1$$

$$a^2 + b^2 = 4\lambda^2 \rightarrow \lambda^2 = \frac{a^2 + b^2}{4} \rightarrow \lambda = \pm \sqrt{\frac{a^2 + b^2}{4}} \rightarrow \lambda = \pm \frac{\sqrt{a^2 + b^2}}{2}$$

$$\boxed{\lambda_1 = +\frac{\sqrt{a^2 + b^2}}{2}} \quad \wedge \quad \boxed{\lambda_2 = -\frac{\sqrt{a^2 + b^2}}{2}}$$

We have two values, which means that we have two stationary points. First, we replace λ_1 :

$$\lambda_1 = +\frac{\sqrt{a^2 + b^2}}{2}$$

$$x = -\frac{a}{2\lambda} \rightarrow x = -\frac{a}{\cancel{2} \frac{\sqrt{a^2 + b^2}}{\cancel{2}}} \rightarrow \boxed{x = -\frac{a}{\sqrt{a^2 + b^2}}}$$

$$y = -\frac{b}{2\lambda} \rightarrow y = -\frac{b}{\cancel{2} \frac{\sqrt{a^2 + b^2}}{\cancel{2}}} \rightarrow \boxed{y = -\frac{b}{\sqrt{a^2 + b^2}}}$$

$$\boxed{M\left(-\frac{a}{\sqrt{a^2 + b^2}}, -\frac{b}{\sqrt{a^2 + b^2}}\right)}$$

Investigate whether the maximum or minimum over the second-order total differential:

$$\frac{\partial F}{\partial x} = a + 2\lambda x$$

$$\frac{\partial F}{\partial y} = a + 2\lambda y$$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

from here we have :

$$d^2 F = \frac{\partial^2 F}{\partial x^2} dx^2 + 2 \frac{\partial^2 F}{\partial x \partial y} dx dy + \frac{\partial^2 F}{\partial y^2} dy^2$$

$$d^2 F = 2\lambda dx^2 + 0 + 2\lambda dy^2$$

$$\boxed{d^2 F = 2\lambda(dx^2 + dy^2)}$$

$$\lambda_1 = + \frac{\sqrt{a^2 + b^2}}{2}$$

$$d^2 F = \cancel{\lambda} \frac{\sqrt{a^2 + b^2}}{\cancel{\lambda}} (dx^2 + dy^2)$$

$$d^2 F = \sqrt{a^2 + b^2} \cdot (dx^2 + dy^2) \rightarrow \boxed{d^2 F > 0}$$

As is $d^2 F > 0$, point is **minimum**, replace in the initial function to find the minimum value:

$$z_{\min} = ax + by$$

$$z_{\min} = a\left(-\frac{a}{\sqrt{a^2 + b^2}}\right) + b\left(-\frac{b}{\sqrt{a^2 + b^2}}\right)$$

$$z_{\min} = -\frac{a^2}{\sqrt{a^2 + b^2}} - \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$z_{\min} = -\frac{a^2 + b^2}{\sqrt{a^2 + b^2}}$$

$$\boxed{z_{\min} = -\sqrt{a^2 + b^2}}$$

Now do the same procedure for another value $\lambda_2 = -\frac{\sqrt{a^2 + b^2}}{2}$:

$$\lambda_2 = -\frac{\sqrt{a^2 + b^2}}{2}$$

$$x = -\frac{a}{2\lambda} \rightarrow x = -\frac{a}{\cancel{\lambda} \frac{\sqrt{a^2 + b^2}}{\cancel{\lambda}}} \rightarrow \boxed{x = \frac{a}{\sqrt{a^2 + b^2}}}$$

$$y = -\frac{b}{2\lambda} \rightarrow y = -\frac{b}{\cancel{\lambda} \frac{\sqrt{a^2 + b^2}}{\cancel{\lambda}}} \rightarrow \boxed{y = \frac{b}{\sqrt{a^2 + b^2}}}$$

$$\boxed{N\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right)}$$

$$\boxed{d^2F = 2\lambda(dx^2 + dy^2)}$$

$$\lambda_2 = -\frac{\sqrt{a^2 + b^2}}{2}$$

$$d^2F = -\cancel{2} \frac{\sqrt{a^2 + b^2}}{\cancel{2}} (dx^2 + dy^2)$$

$$d^2F = -\sqrt{a^2 + b^2} \cdot (dx^2 + dy^2) \rightarrow \boxed{d^2F < 0}$$

We conclude that this is the point of **maximum**, and to determine the maximum function value, we do:

$$z_{\max} = ax + by$$

$$z_{\max} = a \frac{a}{\sqrt{a^2 + b^2}} + b \frac{b}{\sqrt{a^2 + b^2}}$$

$$z_{\max} = \frac{a^2}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$z_{\max} = \frac{a^2 + b^2}{\sqrt{a^2 + b^2}}$$

$$\boxed{z_{\max} = \sqrt{a^2 + b^2}}$$

Example 2.

Find conditional extremes for function $u = x - 2y + 2z$ **if the condition is** $x^2 + y^2 + z^2 = 1$

Solution:

$$F = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$$

$$F = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x$$

$$\frac{\partial F}{\partial y} = -2 + 2\lambda y$$

$$\frac{\partial F}{\partial z} = 2 + 2\lambda z$$

$$1 + 2\lambda x = 0 \rightarrow x = -\frac{1}{2\lambda}$$

$$-2 + 2\lambda y = 0 \rightarrow y = \frac{1}{\lambda}$$

$$2 + 2\lambda z = 0 \rightarrow z = -\frac{1}{\lambda}$$

$$x^2 + y^2 + z^2 = 1$$

$$\left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 = 1$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 1 \dots \dots \dots / \cdot 4\lambda^2$$

$$1 + 4 + 4 = 4\lambda^2$$

$$\lambda^2 = \frac{9}{4} \rightarrow \lambda = \pm \frac{3}{2} \rightarrow \boxed{\lambda_1 = +\frac{3}{2}} \vee \boxed{\lambda_2 = -\frac{3}{2}}$$

Again we have **two values for λ** :

For $\lambda_1 = +\frac{3}{2}$

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x \rightarrow \frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$\frac{\partial F}{\partial y} = -2 + 2\lambda y \rightarrow \frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$$\frac{\partial F}{\partial z} = 2 + 2\lambda z \rightarrow \frac{\partial^2 F}{\partial z^2} = 2\lambda$$

$$d^2F = 2\lambda d^2x + 2\lambda d^2y + 2\lambda d^2z = 2\lambda(d^2x + d^2y + d^2z)$$

$$d^2F = 3 \cdot \frac{3}{2} (d^2x + d^2y + d^2z)$$

$$d^2F = 3(d^2x + d^2y + d^2z) \rightarrow \boxed{d^2F > 0}$$

$M\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ is minimum

$$u = x - 2y + 2z$$

$$u_{\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)} = -\frac{1}{3} - 2 \cdot \frac{2}{3} + 2 \cdot \left(-\frac{2}{3}\right) = -3$$

$$\boxed{u_{\min} = -3}$$

For $\lambda_2 = -\frac{3}{2}$

$$x = -\frac{1}{2\lambda} \rightarrow \boxed{x = \frac{1}{3}}$$

$$y = \frac{1}{\lambda} \rightarrow \boxed{y = -\frac{2}{3}}$$

$$z = -\frac{1}{\lambda} \rightarrow \boxed{z = +\frac{2}{3}}$$

$$\boxed{N\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)}$$

then is

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x \rightarrow \frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$\frac{\partial F}{\partial y} = -2 + 2\lambda y \rightarrow \frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$$\frac{\partial F}{\partial z} = 2 + 2\lambda z \rightarrow \frac{\partial^2 F}{\partial z^2} = 2\lambda$$

$$d^2 F = 2\lambda d^2 x + 2\lambda d^2 y + 2\lambda d^2 z = 2\lambda(d^2 x + d^2 y + d^2 z)$$

$$d^2 F = \cancel{2} \cdot \left(-\frac{3}{\cancel{2}}\right)(d^2 x + d^2 y + d^2 z)$$

$$d^2 F = -3(d^2 x + d^2 y + d^2 z) \rightarrow \boxed{d^2 F < 0}$$

$$N\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \text{ je maximum}$$

$$u = x - 2y + 2z$$

$$u_{\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)} = \frac{1}{3} - 2 \cdot \left(-\frac{2}{3}\right) + 2 \cdot \frac{2}{3} = 3$$

$$\boxed{u_{\max} = 3}$$

Differentiable function $f(x)$ reaches its maximum or minimum value in a closed and limited area, in stationary point or point in the border area.

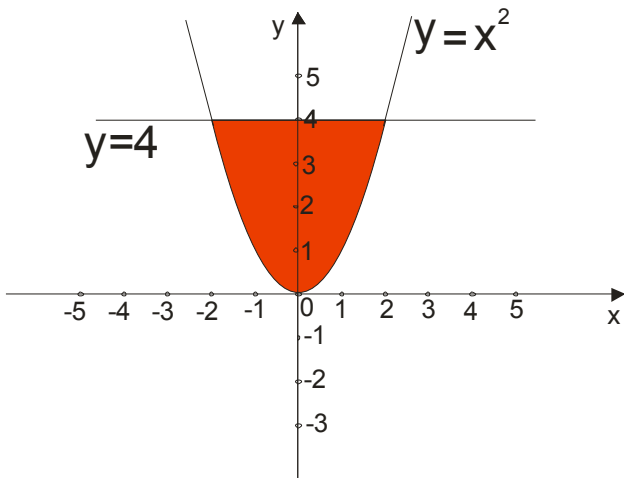
Example 3.

Find the minimum and maximum value for function $z = 2x^3 + 4x^2 + y^2 - 2xy$ in a closed area

limited with curves: $y = x^2$ and $y = 4$

Solution:

Here we must first draw a picture and see the area:



Stationary point we are looking at the casual way:

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$\frac{\partial z}{\partial x} = 6x^2 + 8x - 2y$$

$$\frac{\partial z}{\partial y} = 2y - 2x$$

$$\frac{\partial z}{\partial x} = 0 \rightarrow 6x^2 + 8x - 2y = 0$$

$$\frac{\partial z}{\partial y} = 0 \rightarrow 2y - 2x = 0 \rightarrow \boxed{x = y}$$

$$x = y \text{ replace in } \rightarrow 6x^2 + 8x - 2y = 0$$

$$6x^2 + 8x - 2x = 0 \rightarrow 6x^2 + 6x = 0 \rightarrow x_1 = 0, x_2 = -1$$

$$x_1 = 0 \rightarrow y_1 = 0 \rightarrow \boxed{M_1(0,0)}$$

$$x_2 = -1 \rightarrow y_2 = -1 \rightarrow \boxed{M_2(-1,-1)}$$

Next we must determine the value of initial function in these points:

$$M_1(0,0)$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(0,0)} = 2 \cdot 0^3 + 4 \cdot 0^2 + 0^2 - 2 \cdot 0 \cdot 0 = 0$$

$$\boxed{z_{(0,0)} = 0}$$

and

$$M_2(-1,-1)$$

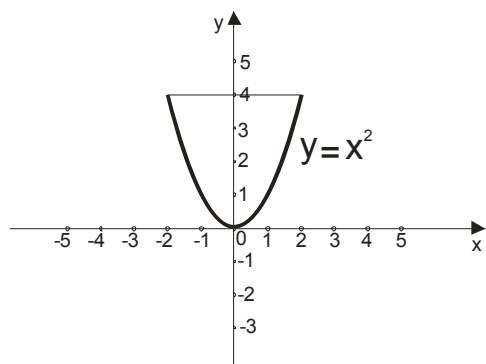
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(-1,-1)} = 2(-1)^3 + 4(-1)^2 + (-1)^2 - 2(-1)(-1) = -2 + 4 + 1 - 2 = 1$$

$$\boxed{z_{(-1,-1)} = 1}$$

For now **we do not know** whether these points are extreme points, we must examine border points.

First, we examine the **point of parable**:



Replace $y = x^2$ in a given function and looking derivate.

$$y = x^2$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(y=x^2)} = 2x^3 + 4x^2 + (x^2)^2 - 2x(x^2)$$

$$z_{(y=x^2)} = \cancel{2x^3} + 4x^2 + x^4 - \cancel{2x^3}$$

$$z_{(y=x^2)} = x^4 + 4x^2$$

derivate:

$$z' = 4x^3 + 8x$$

$$z' = 0 \rightarrow 4x^3 + 8x = 0 \rightarrow 4x(x^2 + 2) = 0 \rightarrow \boxed{x = 0}$$

$$y = x^2 \rightarrow y = 0^2 \rightarrow \boxed{y = 0}$$

We have point $\boxed{M_3(0,0)}$

This point we have already received as stationary ...

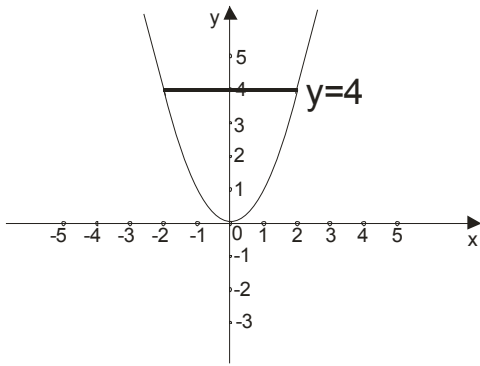
$$M_3(0,0) = M_1(0,0)$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(0,0)} = 2 \cdot 0^3 + 4 \cdot 0^2 + 0^2 - 2 \cdot 0 \cdot 0 = 0$$

$$\boxed{z_{(0,0)} = 0}$$

Now examine the point on the line $y = 4$



$$y = 4$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(y=4)} = 2x^3 + 4x^2 + 4^2 - 2x \cdot 4$$

$$z_{(y=4)} = 2x^3 + 4x^2 - 8x + 16$$

derivate:

$$z' = 6x^2 + 8x - 8$$

$$z' = 0 \rightarrow 6x^2 + 8x - 8 = 0 \rightarrow x_1 = \frac{2}{3}, \quad x_2 = -2$$

$$\boxed{M_4 = \left(\frac{2}{3}, 4\right)}, \quad \boxed{M_5 = (-2, 4)}$$

$$M_4\left(\frac{2}{3}, 4\right)$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{\left(\frac{2}{3}, 4\right)} = 2 \cdot \left(\frac{2}{3}\right)^3 + 4 \cdot \left(\frac{2}{3}\right)^2 + 4^2 - 2 \cdot \left(\frac{2}{3}\right) \cdot 4 = 0 \quad \text{and}$$

$$z_{\left(\frac{2}{3}, 4\right)} = \frac{16}{27} + \frac{16}{9} + 16 - \frac{16}{3} = \frac{176}{27} = 6\frac{12}{27}$$

$$\boxed{z_{\left(\frac{2}{3}, 4\right)} = 6\frac{12}{27}}$$

$$M_5(-2, 4)$$

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$z_{(-2, 4)} = 2 \cdot (-2)^3 + 4 \cdot (-2)^2 + 4^2 - 2 \cdot (-2) \cdot 4 = 0$$

$$z_{(-2, 4)} = -16 + 16 + 16 + 16$$

$$\boxed{z_{(-2, 4)} = 32}$$

So:

The function has a maximum value at point $M_5(-2, 4)$ that is : $z_{(-2, 4)} = 32$

The function has a minimum value at point $M_3(0, 0) = M_1(0, 0)$ that is : $z_{(0, 0)} = 0$